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SYLLOGISTICS, MODALITY, TRIVALENCE

Abstract

Both modal systems and syllogistics are pseudo-Boolean logic cases for which Boolean function class construction (FB) is insufficient. A complete and consistent trivalent computing system can provide semantics for such Boolean logics, the developing strategy following two classes: the existing Boolean functions (FBE) and incomplete or partial (FBI) ones. In the first part of this paperwork we are dealing with a complete trivalent logic axiomatization, taking into consideration a complete class namely the trivalent function class (FT) as well as the closed classes of trivalent functions (in the way of Emil L. Post). In this second part, as we noticed that Venn's diagram model is consistent, complete and non-ambiguous for building the immediate syllogistic inferences and Aristotelian syllogistic, we are trying to approach a modal interpretation of syllogistics. Finally, based on this Boolean semantics provided by syllogistic interpretation, we have in view to build modal computing systems starting with T minimal system.

Keywords: Boolean semantics, modal semantics, modal syllogistics, closed or completed class of functions

1. Modality semantic models and trivalent function classes

All the axiomatic system of a complete propositional calculus enumerated in the following section should be considered as made up with MP rules (modus ponens or detaching rule) and *RS* rules (substitution rule) − thus including a system of *CPI (implicative propositional calculus)*. Beforehand, however, we need a preparatory debate on the syntactic slope and on the semantic one of the construction. Syntax and semantics should make up with the help of two distinct sets: $\boldsymbol{F}_{\{\neg,\, \supset\}}$ for syntax and *FB* for semantics. $\boldsymbol{F}_{\{\neg,\, \supset\}}$ is the set of formulas (well formulated) with the help functors "¬" (negation) and "⊃" (implication) and *FB* is the Booleeen function set. Out of the subsequent construction, we should find out linking elements between the two sets.

If there are not any confusing possibilities, we should denote set $\boldsymbol{F}_{\{\neg,\, \supset\}}$ by $\boldsymbol{F}$. Set $\boldsymbol{F}$ can be systematically built by the help of some iterative rules ($\boldsymbol{V}$ is a set of variables)

i) $p \in \boldsymbol{F}$, where $p \in \boldsymbol{V}$;

ii) if $A, B \in \boldsymbol{F}$, then $(\neg A), (A \supset B) \in \boldsymbol{F}$.

In a formal system with axioms and rules, the deductive process is thus defined. Starting from MP rules (modus ponens) and RS (substitution rule):

MP) $A \supset B, A \vdash B$

RS) if $A, B \in \mathbf{F}, p \in \mathbf{V}$ then of $A(...p...)$ results $A(...B...)$,

where $A(...B...)$ is the substitution result (homogeneous and rigorous of) p variable, contained by formula A, by formula B. The substitution of p by B should be denoted also by B / p.

Further on we should notice by F^n *(with $n = 1, 2, ...$)* set of the formulas (well formed) with n variable built with the help of specified operators and by V^n *(with $n = 1, 2, ...$)* the set of variables $p_0, ..., p_{n-1}$ with the help of which these formulas are built. For classes with a small number of variables we can invoke different letters $p, q, r, ...$ – but finally the construction should be extended for F^n classes with n however large. In fact, in grounding the axiomatic systems the small number class variables are privileged: F^1, F^2, F^3.

On the semantic slope, to each formula $A \in \mathbf{F^n}$ it is uniquely associated a Boolean value $v: B^n \rightarrow B$, where $B = \{ 1, 0 \}$. In this case also, n can be however large, but construction of v can be achieved starting from functions with one or two variables. Beforehand, we should denote by FB^n the set of all the Boolean valuations with n variables (we may call them also Boolean functions, with a well known cardinal:

$FB^n = 2 \wedge 2^n$,

where we used for power also the sign "^" in order not to overlapping the formulas. Any function $f \in FB^n$ is perfectly valued by a succession of 2^n "bits" (with significance $1 = truth$, $0 = false$)

$f^v = \alpha_{2^\wedge n-1}\, \alpha_{2^\wedge n-2} \ldots \alpha_1\, \alpha_0$,

where $\alpha_{2^\wedge n-1} = f(1, ..., 1)$, $\alpha_{2^\wedge n-2} = f(1, ..., 1, 0)$, ..., $\alpha_0 = f(0, ..., 0)$.

For variables also the value rendering of 2^n values begin from *(1, ..., 1) – "1"* taken of n times. So that we should have $p^v = 10\ FB^1$ class; $p^v = 1100$ and $q^v = 1010\ FB^2$ class and so on. Negation and implication have the valuations as follows:

$(\neg p)^v = 01$; $(p \supset q)^v = 1011$,

as soon as the tautology with two variables should be designated by

$(p \circ q)^v = 1111$,

And, by extension, it can be erected the tautology with how many variables (or with a single variable). As we know, tautology describes semantic validty of a formula with n variables ($n \geq 1$). Further on, when there is not the possiblity of confusion, we should invoke a n variable function by succession of the *2^n* bits of the above representation.

Besides the complete function classes FB (Boolean function set) and FT (trivalent function set), of a great importance in making up the set of universallytruth affirmations in a theory is also the closed function classes in the style of Emil L. Post.

According to this theory, FT^n system elementary function should be type $i^n{}_p$, variables that can be represented by length sequences 3^n.. We shall note by E the set of all these functions and by E^n the set of elementary functions of variable n, $n > 0$. Also we shall denote by I^n *identity-function* of the same class, function that, obviously should fulfill the property:

$I^n (x_0, x_1, ..., x_{n-1}). = (x_0, x_1, ..., x_{n-1})$.

The operator of superposition of functions $f, g, h_1, h_2, ..., h_t$, having, respectively, the positive parity $n, t, n, n, ..., n$, should be denoted by $f = Sup (g, h_1, h_2, ..., h_t)$ and should represent the function

$f(\mathbf{x}) = g(h_1(\mathbf{x}), h_2(\mathbf{x}), ..., h_t(\mathbf{x}))$

for any argument of n variable, x , $x = (x_0, x_1, ..., x_{n-1})$ belonging to the common definition domain of functions, $h_1, h_2, ..., h_t$.

Definition. Trivalent set function A is called *A-sequence* if it is made up of a finite function sequence $f_0, f_1, ..., f_s$ with the propriety that $f_i \in E \subseteq A$ for any $i \in \{ 0, 1, ..., s \}$ or f_i is obtained from sequence by superposition of previous terms. The set of Boolean functions that can be inserted in A-sequences is denoted by $\bar{A}$, representing the closing of set A.

Obviously, if $g, h_1, ..., h_t \in A$, theni $Sup(g, h_1, ..., h_t) \in \bar{A}$. Some properties of closing trivalent function sets can immediately proofed[1]: if A and I are trivalent function sets, then:

i) $E \subseteq \bar{A}$; *ii)* $A \subseteq \bar{A}$; *iii)* $\bar{A} \subseteq \bar{I}$; *iv)* if $A \subseteq \bar{I}$, then $\bar{A} \subseteq \bar{I}$;

v) if $A \subseteq I$, then $\bar{A} \subseteq \bar{I}$;*vi)* $-(-A) = \bar{A}$

(where we marked the double closing of horizontal signs).

In other words, the above defined operator is a real closing operator – here can be seen as a finitely-generated adherence, being an expansive operator (*ii*), monotonous (*iii*) and idempotent (*v*). Morover, set A is closed if and only if $E \subset A$ and any function superposition of A is in A.

We shall render bellow more function classes with a closing "behaviour" and should primarily survey them if in form $i^n{}_p$ elementary function classes, the superposition operation preserves the viewed properties. It is obvious that FT, trivalent function set of any variables is closed. We should render further some closed function classes of FT.

[1] Constantin Cazacu, Valeria Slabu, *Logica matematică (Mathematical logic*, Editura "Ştefan Lupaşcu", Iaşi, 1999, pp. 20-25; S. V. Iablonskij; G. P. Gavrilov; V. B. Kudriavţev, *Funkţii alghebry loghiki i klassy Posta*, Izd. Nauka, Moskva, 1966, cap. II, III; S. V. Iablonskij, *Introduction aux mathématiques discrètes*, Edition Mir, Moscou, 1983, pp. 26-37

1. Elementary functions. Elementary function set, E, is closed too. Indeed, functions $g, h_1,..., h_r \in E$ and $f = Sup(g, h_1, ..., h_r)$, with corresponding parities. Then

$f(x_1, ..., x_n) = g(h_1(x_1, ..., x_n), ..., h_l(x_1, ..., x_n)) = h_t(x_1, ..., x_n) = x_t$.

Over elementary function set we should define, in the style of I. G. Rosenberg (Rosenberg, 1977), a relation for describing subsets of functions of FT. A relation h, ρ on set E, with $h \geq 1$, is a subset of E^h whose elements should be written as columns:

$(a_1, ..., a_h)^T \in \rho \Leftrightarrow (a_{1i}, ..., a_{hi})^T \in \rho$ for any i, $1 \leq i \leq h$,

where $a_i = (a_{i1}, ..., a_{in})$. Relation ρ is written as a matrix whose columns are elements of the relation. Then, the function set that preserves ρ relation (denoted by $Pol\ \rho$) is defined by:

$Pol\ \rho = \{ f / (a_1, ..., a_h)^T \in \rho \Rightarrow (a_{1i}, ..., a_{hi})^T \in \rho \}$

2. T_2, T_1 and T_0 functions. We shall also call these functions as *fixed point functions* – we shall remind that they have, respectively, the property:

$f(2, ..., 2) = 2$ or $f(1, ..., 1) = 1$ or $f(0, ..., 0) = 0$.

If we are in class FT^n, *with* $n > 1$, each of these classes contain a number of *3^(3^(n-1) - 2)* functions. With the above denotation, $T_0 = Pol\ (0) = \{f / f(0, ..., 0) = 0\}$, $T_1 = Pol\ (1)$, $T_2 = Pol\ (2)$. However, it is possible that the functions should have also two fixed points and then we can define $T_{01} = Pol\ (01)$, $T_{02} = Pol\ (02)$, $T_{12} = Pol\ (12)$.

3. Class T functions'. Class T' functions (Słupecki functions) are those for which

$T' = Pol(\{ (a, b, c)^T \in E^3 / a = b \text{ or } b = c \text{ or } c = a \})$,

e.g. those functions whose set of values is less comprehensive than $T = \{ 0, 1, 2 \}$ (non-surjective functions). In general, modal operators that shall be defined in the last section of the paper work are Słupecki functions.

4. Auto-dual functions. We shall designate by A the auto-dual function set, and namely the set of those Boolean functions f for which

$f(x_0, ..., x_{n-1}) = f(x_0 \oplus 1, ..., x_{n-1} \oplus 1) \oplus 1$.

Being trivalent, the negation does not have Boolean properties, thus $f \neq \neg\neg f$, *but* $f = \neg\neg\neg f$ Lemmas of auto-duality can be enunciated for class FT^n in accordance with the model of those of FB^n class. With the above notation, set A of auto-dual functions can be also written as:

$A = \mathrm{Pol} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.

5. Monotonous functions. For describing monotony in class FB (but also in FT), we need the following model.

Considering binary n-uple $\alpha = (\alpha_{0,}..........\alpha_{n-1}), \beta = (\beta_n,.........,\beta_{n-1})$ we shall say $\alpha \leq \beta$ if and only if $\alpha_0 \leq \beta_0,.........,\alpha_{n-1} \leq \beta_{n-1}$ (where

$0 \le 0, 0 \le 1, 1 \le 1$). Relation $\alpha \le \beta$ means by definition $\alpha \le \beta$ and $\alpha \ne \beta$. We say that the n-uple $\alpha \; si \beta$ are neighboring by variable x_i if i exists so that $0 \le i \le n-1, \alpha_i \ne \beta_i \; si \alpha_j = \beta_j$ for j = *0, 1, …., i- 1, i+1,….. n-1*.

Based on a slightly deductible condition, the following definition can be enunciate:

Definition. $f \in M^n \Leftrightarrow$ for any variabiable x_i and for any *n*-uples $\alpha \;, \beta$ neighbored by $x_i, \alpha < \beta \Rightarrow f(\alpha) \le f(\beta)$.; we noted by M^n the set of monotones functions of *n* variabiles.

Let's designate by M the set of all monotonous functions of *FB*. It can be shown that *M* is a closed function class. For defining the monotony in trivalent function class, *FT,* we need three orders, not only one. There three classes of monotonous functions in *FT* namely:

M_0 =Pol $\begin{pmatrix} 0\ 1\ 2\ 2\ 2\ 0 \\ 0\ 1\ 2\ 0\ 1\ 1 \end{pmatrix}$, M_1 = Pol $\begin{pmatrix} 0\ 1\ 2\ 0\ 0\ 1 \\ 0\ 1\ 2\ 1\ 2\ 2 \end{pmatrix}$, M_2 = Pol $\begin{pmatrix} 0\ 1\ 2\ 1\ 1\ 2 \\ 0\ 1\ 2\ 2\ 0\ 0 \end{pmatrix}$,

These being the set of monotonous functions observing, respectively, the orders *2 < 0 <1, 0 <1 <2, 1 <2 <0.*

6. Linear functions. The linear Boolen function class is of those which can be expressed in the form of

$f(x_0,\dots, x_{n-1}) = a_0x_0 \oplus \; \dots \oplus a_{n-1}x_{n-1} \oplus x_n$

7. U type functions. With the help of the above notation, U type functions are expressed as follows:

U_0 = Pol $\begin{pmatrix} 0\ 1\ 2\ 1\ 2 \\ 0\ 1\ 2\ 2\ 1 \end{pmatrix}$, U_1 = Pol $\begin{pmatrix} 0\ 1\ 2\ 0\ 2 \\ 0\ 1\ 2\ 2\ 0 \end{pmatrix}$, U_2 = Pol $\begin{pmatrix} 0\ 1\ 2\ 0\ 1 \\ 0\ 1\ 2\ 1\ 0 \end{pmatrix}$.

8. B type functions. There are the following:

B_0 =Pol $\begin{pmatrix} 0\ 1\ 2\ 0\ 1\ 0\ 0\ 2 \\ 0\ 1\ 2\ 1\ 0\ 1\ 2\ 0 \end{pmatrix}$, B_1 = Pol $\begin{pmatrix} 0\ 1\ 2\ 0\ 1\ 1\ 2 \\ 0\ 1\ 2\ 1\ 0\ 2\ 1 \end{pmatrix}$, U_2 = Pol $\begin{pmatrix} 0\ 1\ 2\ 0\ 2\ 1\ 2 \\ 0\ 1\ 2\ 2\ 0\ 2\ 1 \end{pmatrix}$.

Additional details can be found in a monography of Igor Stojmenovič[2]. We should observe only the fact that the set of functions T_0, T_1, T_2, T_{01}, T_{12}, T_{02}, T'', *A*, *L*, M_0, M_1, M_2, U_0, U_1, U_2, B_0, B_1, B_2, are two by two distinct and should enunciate without demonstration

Theorem 1[3]**.** *FT* class has exactly 18 maximal classes of closed functions and namely T_0, T_1, T_2, T_{01}, T_{12}, T_{02}, T'', *A*, *L*, M_0, M_1, M_2, U_0, U_1, U_2, B_0, B_1, B_2. □

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[2] Igor Stojmenovič, *Some Combinatorial and Algorithmic Problems in Many-Valued Logics*, Editura Universității, Novi Sad, 1988, p. 71

[3] Constantin Cazacu; Valeria Slabu, op. cit., pp. 20-25; S. V. Iablonskij; G. P. Gavrilov; V. B. Kudriavțev, *Funkții alghebry loghiki i klassy Posta*, Izd. Nauka, Moskva, 1966, cap. II, III; S. V. Iablonskij, *Introduction aux mathématiques discrètes*, Edition Mir, Moscou, 1983, pp. 26-37

2. A modal approach of syllogistics

The logic hexade of the categorical reasoning built over the set *{ SuP, SaP, SeP, SiP, SoP*, *SyP, }*, with $SuP = SaP \vee SeP$ şi $SyP = SiP \vee SoP$ is largely debated in literature[4] starting from the idea of Robert Blanché to extend the logic square of Boethius with two more vertices.

We shall represent and list the six categorical reasoning of the logical:

S	*P*	*SaP*	*SeP*	*SiP*	*SoP*	*SuP*	*SyP*
2	*2*	*1*	*0*	*2*	*1*	*0*	*2*
2	*0*	*0*	*1*	*1*	*2*	*0*	*2*
0	*2*	*1*	*1*	*1*	*1*	*1*	*1*
0	*0*	*1*	*1*	*1*	*1*	*1*	*1,*

with clear-cut significances: *0* = "hachure", *2* = "dot" ("asterisk"), *1* = "white, undetermined".

In this way, we should have prefix-representations:

SaP = 1011 (S, P); SeP = 0111 (S, P);

SiP = 2111 (S, P); SoP = 1211 (S, P);

$SuP = 0011(S, P);\ SyP = \underline{22}11\ (S, P)$,

where we should immediately explain the representation by underlining from *SyP* by hachure and dot rules significance.

Indeed just with these two rules of dot *(R2)* and of hachure *(R0)*, we can build both valid mediate inferences (opposition and eductions) and Aristotelian syllogistics[5]; an operational rule *(RO)* is added to these two, providing the conjunction significance from the premises and disjunction from conclusions.

R0) when adding a new variable, hachure is unconditionally "extended" over all available areas;

R2) when adding a new variable, the dot is conditionally "extended" over all available areas by the way of line; more over, in the variable area where only a simple portion remained non-hachured a dot is placed;

RO) the supplementary premise is added by overlapping and the additional conclusion, by the way of line of the area are compatible ("dot" and "hachure" are incompatible.

In the approached reasoning should by in the following form *(Fig. 1)*:

[3] Igor Stojmenovič, op. cit., pp. 90 sqq.

[4] Petru Ioan, *Logica integrală (Integral logic)*, vol. I, Editura „Ştefan Lupaşcu", Iaşi, 1999, pp. 179-234; I. Didilescu, P. Botezatu, *Silogistica. Teoria clasică şi interpretările moderne (Syllogistic. Classic theory and modern interpretations),* Editura Didactică şi Pedagogică, Bucureşti, 1978, pp. 64-78

[5] George Ceauşu, *Logica mirabilis*, Editura Alfa Iaşi, pp. 97-106

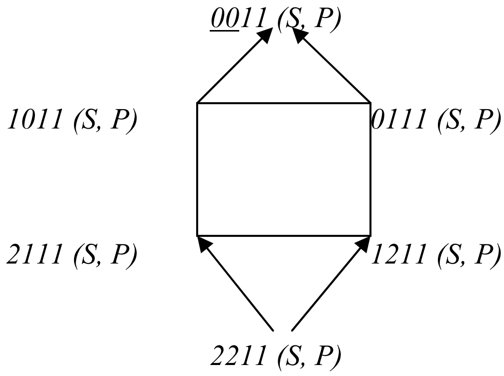

Fig. 1. Logical hexade of syllogistic reasoning

By above representations we described Venn diagrams in terms of a trivalent logic (precisely pseudo-Boolean) with values 2 *("truth"), 0 ("false")* and *1 (undetermined, "white")*.

Further on we should define three kinds of operations for Venn diagrams:

1) complementarity: for the areas not restricted to terms, $\neg 1 = 0$, $\neg 0 = 1$, therefore only areas with hachure and dots are denied and the white is insensitive to negation; for a certain term, the complementarity is carried ot in the way of considering its external area;

2) identity of two Venn diagrams takes place when *"0"*, *"1"* or *"2"* areas are identically placed;

3) sub-alternation of a Venn diagram with another one is translated in an automatic plane by relations

$0 \leq 1 \,;\; 1 \leq 2 \,, a \leq a,$

For each of trivalent a values. In other words, in order to get from a Venn diagram a superordinate one we should suppress the hachure of an area or add the dot on an area (white); by this latter relation we are not able to transform the hachure into dots and reciprocal. The parallelism of fundamental logic categories (notion and sentence), the complementarity translates in a notional plane the equivalence and subordination transposes implication.

Based on classical representation of syllogistic reasoning by Venn diagrams and of above defined three operation, a consistent and complete model can be described for inferences with classical categorical reasoning avoiding didactically difficult explicative models, getting also a representation basis in order to approach the poly-syllogistic issue.

The categorical reasoning hexade, highlighted in *Fig. 1*, needed also the *SuP* and *SyP* reasoning representations by Venn diagrams. When $S = P$, the semantics of these diagrams can go over two variants showed in *Fig. 2 and 3*. In another paper, we described how difficult was to pass from class FT^2 to FT^1 (which are present in our current case). Legitimacy of representations of the two figures bellow is coming out from the same above regulations applied to Venn diagrams applied to a single variable:

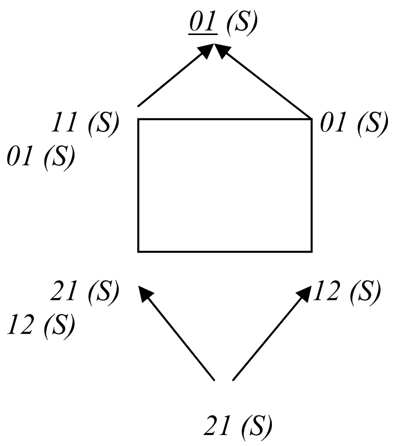

Fig. 2. Logic hexade of syllogistic reasoning in *S* = *P*

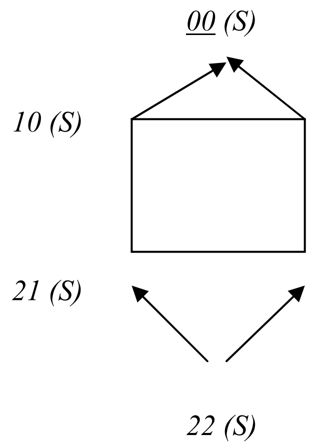

Fig. 3. Logic hexade of syllogistic reasoning in *S* = *P* (another variant)

We can make use of more methods in order to deduct the valid syllogistic modes by John Venn diagrams; for deduction of the invalid ones also from these, so that they represent a consistent and complete model of syllogistics. Moreover, the model is non-ambiguous unlink to the types of representations of syllogistic reasoning (Leibniz-Euler type circular diagrams, the arithmetic-algebraic method of Fred Sommers etc.)[6]. We can proceed to extend the model to other types of categorical reasoning reaching a language diversification of syllogistics – as a basic stage in interpreting the natural language by a predicate calculus.

John R. Searle catalogues in a paper investigating the institution fact, the realistic tendency in science philosophy, setting its following suppositions, among which we specify the first two[7]: *i) world* (reality or universe) exists independently of our representation; *ii)* human beings have access to the world and its features represented in a variety of interconnecting modes by the way of representing category (usually, by intentional procedures). Supposition i) is of an "external realism"

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[6] Petru Ioan, *Curs de logică formalizată 1 (Logic formalized course 1)*, Editura Ştefan Lupaşcu, Iaşi 2004, pp. 24 sqq.

[7] John Searle, *Realitatea ca proiect social (Reality as a social project),* pp. 124-136

and ii) is of a “representational”: the psychic notion of representation with a rather limited meaning within the triangle sensation-perception-representation extends also on the other corners of the triangle, “absorbing” also other categories of psychology. The formalism of these principles could be set by the help of key predicate

representation (x, y, z) = “representation by x of y from z perspective”;

We can notice with this occasion, the realists’ preference for wording with existential suppositions like:

“certain S are P ” = “it is possible that S to be P” *(“possibilist realism”)*;

“certain S are not P ” = “it is possible that S not to be P” *(“contingent realism”)*.

Certain psychic representations are real, other ones fictional – therefore it is noticed that possibility and contingence are minimal features requested for certain (real) assertions. When we shall invoke total notion universe, we shall have another landscape:

“all S are P ” = “it is necessary that S to be P” *(“desiderability”,” utopia”)*;

“all S are not P ” = “it is impossible that S not to be P” *(“impossibility”)*.

The utopian and impossible representations can be also named fictional (if they are related to the real / imaginary tensioned relation) or constrictive one if we are in a deontical logic (of obligation and permission). Let us observe that “the impossibility” we are referring to has not a utilitarian character but defies the human limits. Besides the linking between syllogistics and modality, the above interpretation gives us an idea of “modal realism” applicable in mind philosophy[8].

The above suggested table representation is one of the few valid solutions for poly-syllogistic; Venn diagrams for three terms are expressive enough, but, from four terms above, the planar representation of the four intersected circles becomes difficult. The above suggested representation (e.g., *1011 (S, P)* for *SaP*), inspired by truth logic matrices, could be used for a sufficient large number of terms. For example we could have *01210111 (S, P, M)* – as the logical diagram of Ferio mode looks in figure I for meta-variables (of notions) *S = 22220000 (S, M, P)*; *P = 22002200 (S, M, P)*; *M = 20202020 (S, M, P)*.

We may consider also the representation on n-foils*, (f)*; the representation of this type has a remarkable property: on the 2^n horizontals, the binary number made up of variable values (for which the function is computer) decreases starting from *1*n* with a unit on each line, formally reaching to *0* (more exactly, *0*n*), which means that the *2^n* lines on which the variables take values as well as the

[8] George Ceauşu, *Curs de filosofia minţii (Mind philosophy course),* in: Petru Bejan (coord.), *Filosofie, vol VIII, Învăţământ la distanţă, anul IV semestrul II (Philosphy vol VIII, Distance courses, IV year, II semester)*, Editura Universităţii “Al. I. Cuza”, Iaşi, 2007, pp. 165, 166

corresponding values of Boolean functions should be grouped in n sets, named, on a row, n-foils *0, 1, ..., n*.

The above demonstration is inspired from the representation of the three notional variables as Venn diagrams, for which we have a central area (*trefoil 3*), placed at the intersection of the three variables; three areas placed on trefoil 2 (interior, situated at the intersection of the two variables) and the three on trefoil 1 (exterior, containing the areas belonging to only one variable) and, finally, a last external area (trefoil 0), "the landscape" of the discourse universe.

For example, Venn diagram for four variables could not be figured in the plane anymore: a planar graph cannot be but of some prescribed types, which exclude the existence of a planar graph with 16 areas (which we have to delimit by intersection of the four variables). However, by extension, we may suppose the existence of asset of concatenate variables, namely *{ 2222 }* (*quatrefoil 0*), *{ 2220, 2202, 2022, 0222 }* (*quatrefoil 1*), *{ 2200, 2020, 0220, 2002, 0202, 0022 }* (*quatrefoil 2*), { *2000, 0200, 0020, 0002* } (*quatrefoil 3*), *{ 0000 }* (*quatrefoil 4*). For more than four variables $(n > 4)$ we may take into account the combinatory disintegration of the exponential function 2^n.

3. Modal systems with theses and rules

A basis (complete minimal set) within FB class is defined by Emil L. Post as a function that is: neither of fixed point, nor auto-dual, nor monotonous or linear; it is the so-called "Post's small theorem". "Post's big theorem" states all near 50 minimal classes of closed Boolean functions[9]. In FB^2 class there are only two such functions (incompatibility and rejection) and in FB^3 class there are 56 functions. A general formula for the FB^n class is given in another paper[10]. It is interesting that M. Sheffer type function, like Webb function,

$$g(p, q) = \neg(p \vee q) = 000022021(p, q)$$

may serve for building a complete trivalent calculus but for incomplete calculus it behaves unforeseeable as in the class of FT trivalent functions (unlike of what is happening in FB class) not any of functions admits a base! In other words, though it seems paradoxical, it is easier to build a complete trivalent calculus then a subset representing an incomplete calculus: we may have the surprise to have a countable infinity of basic operators!

Tautologic type functions that we should reach following the valid interpretation construction over FT are rich enough in closing properties, being of T_2

[9] Vezi şi S. V. Iablonskij; G. P. Gavrilov, V. B. Kudreavţev, op. cit. pp. 78 sqq.

[10] George Ceauşu, *Asupra posibilităţii de reducere a operatorilor, noţiunilor şi tezelor din cadrul teoriilor ştiinţifice (One possibility of reduction of operators, notions and theses with the scientific theories)* , în: Ana Gugiuman (coord.), *Idee şi valori perene în ştiinţele socio-umane. Studii şi cercetări. Tom VI (Ideas and perennial values in social-human sciences. Studies and research, Tom VI)*, Editura Argonaut Cluj-Napoca, pp. 105-119

class, linear, monotonic and *T'*. They are not auto-dual (they are still counter-dual) neither of class *U* or *B*. Antilogies (total contradictions) should have the same properties, except the fact that they are functions of T_0 class, not T_2. When the modal operators enter too, the closed functions of the maximal classes should restrict as well which simplify the construction procedure of class tautologies. Based on the *theorem 1* and on modal operator selection (in compliance with the specifications of first section of the paper) the class set of closed function of FT can be restricted.

Next theorem comes out from *theorem 1*, considering the closed maximal classes of Boolean functions *(T_1, T_0, A, L, M)* and particularizing the representation for monary classes (both trivalent and Boolean).

Theorem 2. There are 12 closed maximal classes of monary trivalent functions that are not Boolean:

1) three classes of function with two fixed points, $T_{01}{}^{1}$, $T_{12}{}^{1}$, $T_{20}{}^{1}$ for example T_{12} = *{ 210, 211, 212 }*;

2) two classes of monotony for the "clones" of variables, $M_0{}^1$ and $M_2{}^1$, for example

$M_0{}^1$ = *{ 210, 102; 222, 111, 000; 100, 110, 112, 122, 002, 022 }* (for "clone" *"102"*);

3) non-surjective function class (Słupecki), T'^{1};

4) three classes $U_0{}^1$, $U_1{}^1$, $U_2{}^1$, non-surjective, but defining the constants in the sense of variable, respectively *"102"*, *"210"* and *"021"*, for example $U_1{}^1$ = *{ 210; 222, 111, 000, 221, 211 }* and the other two do the same for "clones";

4) three classes $B_0{}^1$, $B_1{}^1$, $B_2{}^1$, non-surjective, but specifying the constants in the sense of variable (respectively *"102"*, *"210"* and *"021"*) and of only one fixed point, for example, B_1 = { *210; 222, 111, 000, 221, 211 }* and the other two do the same for "clones".

We may clear that, unlike FB^1 class (where variable p could be clearly specified as *"10"* or *"01"*) in class FT^1, where there are six combinatory alternatives for the set *{ 0, 1, 2 }*, variable *p* may be taken in three modes: *"210"*, *"102"* and *"021"* (three degree altern group with an even number of permutations). The other three components of order 3 symmetric group, namely *"120"*, *"012"* and *"201"* (order 3 odd altern group), shall be named, usually, "negations". Among the variables we have to fix (only for the needs of generating all classes of maximal closed function), a main one (in our case *"210"*) and two "clones" (*"102"* and *"021"*).

Demonstration is immediate (on the basis of *theorem 1*).

Corollary. The other six classes of monary trivalent closed maximal functions, that have also a Boolean acception, are $T_0{}^1$, $T_1{}^1$, $T_2{}^1$, $M_1{}^1$, A^1 and L^1.

In the set FBI, the variable is represented by *"20"* and its "clones" are *"12"* and *"01"* (in fact the "original" representation and "clones" are also relative), but in **section** 2 of the work, within Venn diagrams, we worked with variables *S = 2200 (S, P)*, *P = 2020 (S, P)*,

Theorem 3. There are nine closed maximal classes of modal monary functions which are not Boolean:

1) two classes of monotony for "clones" of variable *"20"*, and namely M_0^1,= { *20*, *12*; *22*, *11*, *00*; *10* } (for "clone" *"12"*),

M_2^1 = { *20*, *01*, *22*, *11*, *00* ; *21* } (for "clone" *"01"*),

3) class of non-surjective functions (Słupecki), T^1 = { *20*, *02*; *21*, *12*; *10*, *01*; *22*; *11*; *00* };

4) three classes U_0^1, U_1^1, U_2^1, non-surjective, but defining the constants in the sense of a variable, respectively *"12"*, *"20"* and *"01"*; for example, U_1^1 = { *20*; *22*, *11*, *00*, *10*, *01* } and the other two do the same for "clones"

4) three classes B_0^1, B_1^1, B_2^1, non-surjective, specifying the constants in the sense of variable (respectively *"12"*, *"20"* and *"01"*) and of only one fixed point, for example, B_1 = { *20*; *22*, *11*, *00*, *21*, *20* } and the other two do the same for "clone"

We achieved a notation abuse as well: in the previous theorem we have classes of trivalent functions, and in this theorem, classes of incomplete Boolean function of the type $f: B^n \to T$, with $B = \{ 0, 1\}$ and $T = \{ 0, 1, 2\}$.

Demonstration is also immediate.

Corollary. The other six classes with modal closed maximal functions (monary) that have also Boolean acception are also T_0^1, T_1^1, T_2^1, M_1^1, A^1 and L^1. Only the three classes of functions with two fixed points are missing.

Concluding, FBI classes are "fine" enough to cover a large "beach" of modal closed maximal functions. For understanding we insert a table with the number of trivalent functions compared to the incomplete Boolean ones:

n	FT^n	FBI^n
1	3^3	3^2
2	3^9	3^4
3	3^{27}	3^8
4	3^{81}	3^{16}

Fig. 4. Table with number of corresponding functions to the two compared classes *FT* and *FBI*

Corroborating the results so far, we can state the following theorem (presented without demonstration):

Theorem 4. All the nine modal specific closed maximal classes stated in the previous theorem contain also the tautology "22", therefore their component functions may be part of any incomplete system of the modal calculus. Of the other six classes, only three of them contain this tautology *(T_2^1, L^1, M^1)*, the previous observation being valid for them too.

Therefore it is suggested a way of construction for increasingly extending modal systems, starting from the minimal system T. On the other hand, George Boolos is suggesting a modal system and more restricted than T, and namely *K*[11], a calculus leading to a more extended system, G, which in a standard model built over a Peano formal arithmetic, leading towards inconsistency phenomena. Not only for Peano formal arithmetic, but also even for certain modal systems the deductive means of Z theory are not suffice for the theory to demonstrate its own consistency.

In the conditions in which the additive arithmetic of Martin Presburger is complete, it seems that multiplication largely "multiplies" the liberties of function and arithmetic formula codification, leading to large classes of indecidable formulas. We can follow, in its consequences also the "pragmatic paradox" remarked by Constantin Cazacu and Valeria Slabu[12]: the additive arithmetic is complete, but practically unusable (because the laborious codification and great disproportionate length of the formulas containing the multiplication), as long as the additive-multiplicative arithmetic, in spite of the incompleteness, admits very elegant codifications of demonstrations (including bijectives!) and convenient representations on computer.

In this case, an interesting conjecture would refer to an incomplete minimal logic system in an axiomatic relation. Such a system is produced in the modal logics by "possible world semantics", topics of modal theory introduced in debate by Clarence L. Lewis, but set in the philosophy of science by Saul Kripke[13]. Essentially, "possible world semantics" of the modal logics (to which the famous logician had a main contribution) provided an argument in sustaining an anti-realistic current in philosophy of science, but the mathematical construction of discourse universe do not have to be assimilated by all means to "genesis of some new worlds" but to axiomizing forms of naïve set theory.

On another location[14] we specified the fact that the axiomatic theories of sets (Zermelo-Fraenkel, Gödel-Bernays, Morse etc.) are building up different acceptations of the idea of set. In other words, Jakko Hintikka (1998) made distinction between "frame" type modules proposed by Stig Kanger and Saul Kripke, aiming at the providing the first one with a standard semantics inspired by a standard arithmetic (Alfred Tarski) and the latter with a non-standard semantics (Leon Henkin). In the latter semantics type, we can identify also forms of axiomatic in-

[11] George Boolos, *The Unprovability of Consistency*, University Press, Cambridge, 1979, p. 23, 24

[12] Constantin Cazacu and Valeria Slabu, op. cit., pp. 216-227

[13] Saul Kripke, *Numire şi necessitate (Denomination and necessity)*, translation by Mircea Dumitru, Editura All Educational, Bucureşti, 2001, pp. 10 sqq.

[14] George Ceauşu, *Logica mirabilis*, pp. 36-52

completeness. The trivalent propositional calculus (for example in the form presented in *proposition* 3) being considered complete too, a standard model, provided with a modulo 3 formal arithmetic, would not lead to inconsistency phenomena. The outcome can be deduced from completeness theory of Presburger formal arithmetic. That means that the incompleteness results for modal logics come only in non-standard models of structure-model type of Saul Kripke.

Let's remind that in a Boolean calculus, when we have only three classes containing the tautology "11", namely T_1, L, M, the process of functional "completion" of the calculus is much simpler that in a trivalent calculus. We would notice the decisive role the successor-function has in arithmetic development – based on the structure of commutative body over the classes of modulo remains n $(n > 1)$ or over natural number set. For various polyvalent logics with k values (k ≥ 2), the successor function serves to build the negation – a Boolean negation for FB set, a trivalent one for FT set and so on. In the formal axiomatic of Giuseppe Peano, the neutral element for addition, 0 and the neutral one for multiplication, 1, become starting element (smallest), respectively, a first iterate of successor function, $1 = 0'$, after which the complete induction principle allows us to build any natural number of its predecesor.

Interpretation is built in closed class theory of Boolean functions by Emil. L. Post worked out since 1941[15]. However, a complete bivalent propositional calculus can be interpreted as Peano arithmetics (in making "successor function" to be actually negation). As set $\{ \neg, \wedge \}$ makes up a based of Boolean functions, the set $\{ 0, 1, \neg, \vee\vee, \wedge \}$ makes up, more over, a closed class of Boolean classes and in the previous set, the negation acts as a successor-function because:

$$\neg p \equiv_{df} p \vee\vee 1 \equiv p \oplus 1;$$

The addition is just the exclusive disjunction $(\vee\vee)$ and modulo 2 multiplication, just the conjunction $(\wedge)$. Also the set of Boolean algebraic operations, namely $\{ 0, 1, \neg, \vee, \wedge \}$ could be subjected to a "hermeneutics" similar to the above one, with the only change that not exclusive disjunction, but the disjunction $(\vee)$ would be a kind of addition. A bivalent CP is consistent and complete too in a restrained sense. A set of axioms for such a complete calculus was provided in other situations making use of the special constant f, $\neg p \equiv_{df} p \Rightarrow f$.

In other words, the intuitionist systems take full advantage of presence of sign "$\perp$" or f in working up the Heyting algebras that serve them as a support.

The consistence and completeness, two meta-systemic properties largely studied of the complete Boolean propositional calculus, can be expressed in the form of

$$Con)\ \forall A,\ \forall I, I \mid\!- \neg(A \wedge \neg A)\ ;\ Com)\ \forall A,\ \forall I, I \mid\!- (A \vee \neg A),$$

[15] Emil L. Post, *Two-valued Iterative Systems*, 1941

Where we see “disguised” the two secular principles of logics, non-contradiction principle $(\neg (A \wedge \neg A)$ and of excluded tertiary principle $(A \vee \neg A)$ in the above relation, *“A”* is formula, *“I”*, set of hypotheses and the sign *“|-”* marks the presence of a deductive tree between the set of hypotheses A and formula I. The form susceptible of generality of these properties is, actually:

$Con)\ \forall A,\ \forall I, I\ |\!-\ (A \wedge (A \oplus 1)) \oplus 1\ ;\ Com)\ \forall A,\ \forall I, I\ |\!-\ (A \vee (A \oplus 1)),$

as $\neg A = A \oplus 1$.

For a trivalent propositional calculus, the above formulas become:

$Con3)\ \forall A,\ \forall I, I\ |\!-\ (A \wedge (A \oplus 1) \wedge (A \oplus 2)) \oplus 1;$

$Com3)\ \forall A,\ \forall I, I\ |\!-\ A \vee (A \oplus 1) \vee (A \oplus 2),$

In fact, being, the consistency principle for trivalent case, respectively excluded quart principle. The form of the principles can be explained in case of the last calculus due to the existence of two negations, namely $p \oplus 1$ and $p \oplus 2$, where the logic operator *“⊕”* is, in fact, modulo *3* addition. No doubt, the Boolean negation also can be expressed in the form $p \oplus 1$, where the operator is, this time, modulo 2 addition, so that *Con3* and *Com3* formulas are similar to *Con* and *Com*.

The actual principles may be soon generalized for a polyvalent calculus with *p* true values ($p \geq 3$, *p* being prime number):

$Conp)\ \forall A,\ \forall I, I\ |\!-\ (A \wedge (A \oplus 1) \wedge \dots \wedge (A \oplus (p-1))) \oplus 1;$

$Comp)\ \forall A,\ \forall I, I\ |\!-\ A \vee (A \oplus 1) \vee \dots \vee (A \oplus (p-1)).$

In a polyvalent calculus with *p* true values, the two possible negations *p-2* are $A \oplus 1, \dots A \oplus (p-1)$.

Extension to the case of a k-valent logics is ranging on the same line[16]. If number k of the truth values of the polyvalent calculus is not a prime number, the two principles – non-contradiction and completeness ones – can be stated with the help of well known function φ of Leonard Euler:

$Conn)\ \forall A,\ \forall I, I\ |\!-\ (A \wedge (A \oplus 1) \wedge \dots \wedge (A \oplus \varphi(k))) \oplus 1;$

$Comn)\ \forall A,\ \forall I, I\ |\!-\ A \vee (A \oplus 1) \vee \dots \vee (A \oplus \varphi(k)),$

$\varphi(k)$ being the number of the prime numbers with n from the interval *[1, k-1]*.

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[16] George Ceauşu, op. cit., p. 79

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