FUZZY NUMBERS, GAME THEORY AND CLASSIFICATION ALGORITHMS AS AN AID TO SUSTAINABLE CARE FOR THE ELDERLY

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Abstract

One of the biggest problems in the modern world, especially in the European Union, is the medical and social care of the elderly. Furthermore, by increasing the average age and the culture of individuals, an enhancement of the elderly as a resource is required. In this work, after a description of some of the rights and needs of the elderly, we try to understand if some mathematical tools can be useful to describe the various situations and to provide useful algorithms to make decisions. The role of *fuzzy numbers* and *game theory* to describe certain aspects is highlighted. The mathematical concepts of *covering* and *fuzzy partition* are used both for a classification of the elderly, and for a classification of the territory for a better organization of services.

Keywords: sustainable care for the elderly, fuzzy numbers, game theory, fuzzy classification algorithms

Résumé

L'aide médicale et sociale aux personnes âgées est l'un des problèmes majeurs du monde moderne, en particulier dans l'Union européenne. En outre, en augmentant l'âge moyen et la culture des individus, il est nécessaire de renforcer les personnes âgées en tant que ressource. Dans ce travail, après une description de certains des droits et des besoins des personnes âgées, nous essayons de comprendre si certains outils mathématiques peuvent être utiles pour décrire les différentes situations et fournir des algorithmes utiles pour prendre des décisions. Le rôle des *nombres fuzzy* et de la *théorie des jeux* pour décrire certains aspects est mis en évidence. Les concepts mathématiques de la *couverture* et de la *partition fuzzy* sont utilisés à la fois pour une classification des personnes âgées et pour une classification du territoire pour une meilleure organisation des services.

Mots-clés: soins durables pour les personnes âgées, nombres fuzzy, théorie des jeux, algorithmes de classification fuzzy

Rezumat

Una dintre cele mai mari probleme din lumea modernă, în special în Uniunea Europeană, este îngrijirea medicală și socială a persoanelor în vârstă. Mai mult, prin creșterea speranţei de viaţă și a nivelului de educaţie al indivizilor este necesară o creștere a vârstei de pensionare ca resursă de dezvoltare socială. În această lucrare, după o descriere a unora dintre drepturile și nevoile persoanelor în vârstă, încercăm să înțelegem dacă unele instrumente matematice pot fi utile pentru a descrie diferitele situații și pentru a oferi algoritmi folositori în luarea de decizii. Este evidențiat rolul numerelor fuzzy și al teoriei

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jocurilor pentru a descrie anumite aspecte. Conceptele matematice de acoperire și partiția fuzzy sunt folosite atât pentru clasificarea vârstnicilor, cât și pentru clasificarea teritoriului pentru o mai bună organizare a serviciilor.

Cuvinte cheie: îngrijire durabilă pentru vârstnici, numere fuzzy, teoria jocurilor, algoritmi de clasificare fuzzy

1. The rights of the elderly and some assistance policies in Italy

In Italy there are more than twelve million citizens included in the elderly; the "life expectancy" of the population doubled over the last century, attesting average life around 76 for men and 81 for women at the beginning of the twenty-first century.

The number of people over sixty-five, considered as the elderly population, has progressively increased compared to the overall population, from 9.5% in 1961 to 15.3% in 1991; over the next 25 years this number has exceed 23% of the total population.

The need to rethink the role of the elderly in society is now widespread: it is in the general interest to recover the time of active seniority as a useful time for society, considering the elderly as a "resource" essential to the growth and development of the whole society.

For some years in Italy the elderly obtained to be a "safeguarded" category: since 1995 there is the "Charter of the rights of the elderly".

In particular, we mention the following articles:

Art. 1 - Right of the elderly to access the "total quality" of human life in which the substance of the common good consists.

Art. 2 - Right to the maintenance of the personal conditions of the elderly to the highest possible degree of self sufficiency on the mental, psychic and physical level.

Art. 8 - Right to have guaranteed an income that allows not only the mere survival, but the continuation of a normal social life integrated in his context and the right to self-determination and self-promotion.

Art. 9 - Right to the potential, resources and personal experiences of the elderly to be valued and used for the benefit of the common good.

Among the basic needs of the elderly there is that of not being left alone in the management of own life. The model of "family assistance" for the elderly is now disappearing. A new model of assistance is the "Elderly Objective Project", since 1992. The services of this model of assistance include:

Integrated Homecare Assistance. Assistance service in the home of the elderly, that varies from interventions exclusively of a social type, (cleaning of the apartment, sending hot meals, psychological support, handling of administrative procedures, etc.) to mixed social and health interventions (nursing assistance, rehabilitation activities, podiatrist interventions, etc.).

Home Hospitalization. It is related to a series of diagnostic and therapeutic interventions, normally usable in hospital, carried out in the place of life of the elderly patient and possibly integrated, for special services, by a brief presence in the healthcare facility with access and transport facilitated. This type of service has undoubted psychological advantages, but requires the presence of family members available as well as preparations with short specific courses, as well as the assignment of equipment (oxygen dispensers, drip planes, respirators, etc.), walking aids and physiological functions, a privileged telephone connection with the hospital.

Admissions in Health Care Residences (HCR). The HCR are defined as extra-hospital structures aimed at providing accommodation, health, care and recovery services to elderly people who are mainly not self-sufficient. The prerequisite for the use of the HCR is the proven lack of suitable family support that allows providing continuous medical treatment and necessary assistance at home.

The current supply of services, although quite composite, often shows little clarity, requires a complex bureaucracy and above all presents, generally, high costs.

2. Fuzzy numbers and fuzzy classifications of the elderly

2.1. *The concept of fuzzy number*

Recall (see, e.g, Klir and Yuan 1995; Maturo and Tofan 2001; Ross 1997; Zadeh 1965; Franchino et al. 2004) that a *fuzzy number* is a function $u: R \to [0, 1]$ with the following properties:

(NF1) $\exists a, b \in R, a \leq b$: $u(x) = 0$ in $R - [a, b]$ and $u(x) > 0$ in (a, b);

(NF2) $\exists c, d \in R, c \leq d$: $u(x) = 1, \forall x \in [c, d]$;

(NF3) $\forall r \in (0, 1)$, the set $\{x \in [a, b]: u(x) \geq r\}$ is a closed interval.

The interval [a, b] is called the *support* of u, indicated by S(u). The elements a and b are the endpoints, left and right, respectively, of u. The interval [c, d] is the *core* of u, indicated by C(u). The fuzzy number u is *simple* if c = d, i.e. the core reduces to a point.

We write $u \sim (a, c, d, b)$ to indicate that u is a fuzzy number with support [a, b] and with core [c, d]. For every r such that $0 \leq r \leq 1$ the interval $\{x \in [a, b]: u(x) \geq r\}$ is noted $[u]^r$ and is said to be the r-*cut* of u. The left and right endpoints, of $[u]^r$, are noted $u_\lambda{}^r$ and $u_\rho{}^r$, respectively. In particular:

- $[u]^0 = [u_\lambda^0 = a, u_\rho^0 = b]$ is the *closure of the support* of u,
- $[u]^1 = [u_\lambda^1 = c, u_\rho^1 = d]$ is the *core* of u.

A fuzzy number u ~ (a, c, d, b) is said to be:

- *continuous* if u(x) is a continuous function;
- *trapezoidal*, denoted u = (a, c, d, b), if:
  - $\forall x \in [a, c), a < c \Rightarrow u(x) = (x-a)/(c-a)$, (3.1)
  - $\forall x \in (d, b], d < b \Rightarrow u(x) = (b-x)/(b-d)$. (3.2)

A simple trapezoidal fuzzy number u = (a, c, c, b) is said to be a *triangular fuzzy number* and usually we write u = (a, c, b).

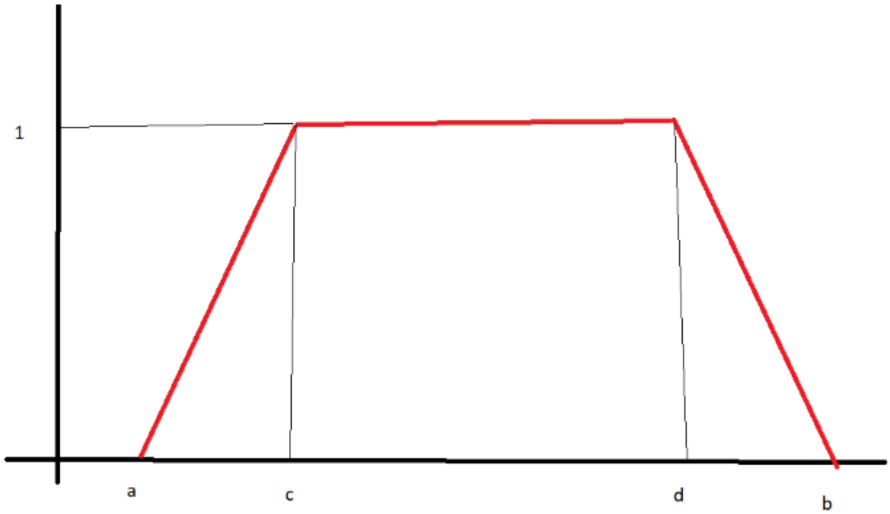

Fig. 1. A trapezoidal fuzzy number

2.2. *Ordering fuzzy numbers*

If [e, f] and [g, h] are two intervals of R, we put

$[e, f] \leq [g, h]$ if $(e \leq g, f \leq h)$.

Moreover, we denote with Φ, T, Δ the set of all the fuzzy numbers, of trapezoidal fuzzy numbers and the triangular fuzzy numbers, respectively.

We have the following propositions:

Proposition 3.1. The relation $\lesssim_M$ such that:

$$\forall u, v \in \Phi, u \lesssim_M v \Leftrightarrow \forall r \in [0, 1], [u]^r \leq [v]^r \text{ (3.3)}$$

is a *partial order relation* on Φ, called the *main order*.

The main order is the basic ordering in the set of fuzzy numbers whatever the shape.

Proposition 3.2. The relation $\lesssim_T$ such that:

$$\forall u, v \in \Phi,\ u \lesssim_T v \Leftrightarrow [u]^0 \leq [v]^0,\ [u]^1 \leq [v]^1 \quad (3.4)$$

is a *partial preorder relation* on Φ, called the *trapezoidal order*.
The restriction of $\lesssim_T$ to the set T of the trapezoidal fuzzy numbers is a *partial order* relation. Such a relation is mainly useful if the trapezoidal shape is preferred, because of the relative simplicity in handling these numbers.

Proposition 3.3. The relation $\lesssim_C$ such that:

$$\forall u, v \in \Phi,\ u \lesssim_C v \Leftrightarrow [u]^1 \leq [v]^1 \quad (3.5)$$

is a *partial preorder relation* on Φ, called the *core order* or *crisp order*.
The restriction of $\lesssim_C$ to the set of simple fuzzy numbers is a *total preorder relation*. The relation $\lesssim_C$ is useful when peripheral spreads are considered of marginal importance with respect to the central ones.

2.3. *The classes of elderly represented by fuzzy numbers*

Analyzing the concept of elder, it seems essential to observe that it is of the fuzzy type and that it is necessary to divide the population of a certain age into classes, each characterized by a particular type of assistance. Each of these classes is represented by a fuzzy number.

Taking into account that the first slight physical problems due to age usually appear after age 55 and that the current law provides for retirement at 65, we believe it is plausible that the entry of an individual into the world of the elderly takes place gradually from 55 to 65 years.

We can also admit that the elderly have an age belonging to the age interval [55, 100]. It does not seem necessary to consider the ages above 100 years, since it is rare for an individual to exceed this age and therefore the number of individuals over 100 years is not relevant from the quantitative point of view.

On the other hand, if, with the increase in well-being and the level of assistance, there will be many centenarians, one can substitute 100 with a higher number.

For example, from the qualitative point of view the elderly can be divided, roughly, into three age classes:

- class C_1: *slightly elderly individuals*, for whom assistance is essentially of a preventive nature, with the need for social aggregation and a little help in cleaning the house;

- class C_2: *self-sufficient elderly individuals*, for whom medical and nursing assistance is important, but who manage to remain self-sufficient for daily needs;
- class C_3: *highly elderly individuals*, for whom due to physical or mental illness, continuous assistance is required.

From the quantitative point of view the class C_i, i =1, 2, 3 is represented by a fuzzy number u_i with the condition $(i < j) \Rightarrow (u_i < u_j)$, with respect to one of the previous order relations. For example, they may be the three trapezoidal fuzzy numbers:

- Class $C_1 \rightarrow$ trapezoidal fuzzy number $u_1 = (55, 65, 70, 75)$,
- Class $C_2 \rightarrow$ trapezoidal fuzzy number $u_2 = (70, 80, 85, 90)$,
- Class $C_3 \rightarrow$ trapezoidal fuzzy number $u_3 = (80, 90, 100, 100)$.

For each age $x \in [55, 100]$, the numbers $u_1(x)$, $u_2(x)$, $u_3(x)$ can indicate, on average, the fractions of individuals for whom a class C_1, C_2, C_3 assistance is required (or is planned), respectively.

2.4. *The different types of classification*

Suppose that individuals of the age belonging to the interval [55, 100] are divided into n classes C_i, represented by fuzzy numbers u_i, with the condition $(i < j) \Rightarrow (u_i < u_j)$, with respect to one of the previous order relations.

Each class C_i, of order i, is characterized by a level of assistance that grows with the growth of the class order. The set $\{u_i, i = 1, 2, \ldots, n\}$ is said to be a *fuzzy classification* of elderly.

For every $x \in [55, 100]$, the number $u_i(x)$ is the degree of assistance to the level i, expected on average, for an individual aged x.

The overall assistance for this individual is the sum:

$$u(x) = u_1(x) + u_2(x) + \ldots + u_n(x). \quad (3.6)$$

The fuzzy classification $\{u_i, i = 1, 2, \ldots, n\}$ is said to be:

(1) *a fuzzy covering*, if $u(x) > 0$, $\forall x \in [55, 100]$,

(2) *a fuzzy partition*, if $u(x) = 1$, $\forall x \in [55, 100]$,

(3) *with lower level* $\alpha > 0$ *covering*, if $u(x) \geq \alpha$, $\forall x \in [55, 100]$,

(4) *with upper level* $\beta > 0$ *covering*, if $u(x) \leq \beta$, $\forall x \in [55, 100]$.

The fuzzy covering condition is the minimal condition of global assistance, while the fuzzy partition appears to be the most correct.

In general, it is to be expected that there are values α, β, with $0 < \alpha \leq \beta \leq 1$ such that the classification has lower level α and upper level β. The thresholds α depend on economic factors and the choices of administrators: the higher the α value, the more a minimum amount of assistance is guaranteed. The higher the β value, the more appropriate assistance is to be expected.

3. Game theory as a tool for an analysis of policies for the elderly

A second aspect to keep in mind is the nature of game with n persons of the system of offer-fruition of services for the elderly.

One of the players is the *public administration* (PA), which decides the strategies to be followed. Often the PA changes the strategies with legislative instruments, according to its objectives. Among the objectives of the PA is not only to *maximize the level of assistance*, but above all there is the need to *minimize costs*.

Another player is the *elder* or a *family member* who represents him and makes his decisions according to the family budget, the urgent needs, the bureaucratic imposition by the PA, the time available and so on.

A third player is the *family doctor*. His preparation, his availability and his knowledge of the behavior of the PA influence his decisions and therefore the usefulness that the elderly can draw.

A fourth player is the *private entity* that manages social and health services, which bases its strategies on the difficulty of the relationship between the elderly and the PA, saving time, minimizing the bureaucracy.

There are other players, such as cooperatives, ONLUS, charities and so on. The effect of such players may be taken into consideration as a random variable or assuming that the benefits of the four fundamental players are fuzzy numbers.

The difficulty for the elder or his representative to make "*rational decisions*" in this four-person game is due to the vast incompleteness of the information on the rules of the game and their continuous changes, so a priority is that the elder is assisted by a managerial figure of trust, who has the information and knows the rules necessary for rational choices. The "*rational decision*" requires a knowledge of the objectives of the elderly, the possible alternatives and the value that each consequence of each alternative takes, in line with these objectives. The elderly person is often influenced by emotional factors, beliefs, prejudices and only an experienced managerial figure, who knows the elderly well, and is able to understand the objectives and situation of the elderly, can adequately assess the various aspects of the decision-making process.

Of course, rationality is limited (Simon, 1982, March, 1994) for various aspects, such as an incomplete understanding of the objectives of the elderly, contradictory objectives, inadequate knowledge or analysis of possible alternatives, inconsistencies of various kinds in assigning scores to the various consequences and so on. In the case of decisions under conditions of uncertainty it is also necessary to assign probabilities, often subjective, to the various possible consequences and calculate the expected utility for each consequence (de Finetti 1974; Lindley 1985; Coletti and Scozzafava 2002).

From the theory of rational decision, the alternative that maximizes expected utility is preferred. Often, however, rather than expected utility, the expected value of each alternative is calculated as the average of the scores weighed with the respective probabilities; the acceptance of these expected values as a measure of the preferability of the alternatives is a further limitation to the rationality of the choices.

In modeling with game theory, it is also necessary to take into account the possible coalitions between the various players and to evaluate the usefulness of cooperation between the various players, possibly calculating the *characteristic function* of the game.

The fuzzy extension of the decision-making process is based on the assignment of fuzzy numbers, usually triangular or trapezoidal, as scores of the alternatives in each of the states of nature, or as subjective probabilities, so as to take into account the uncertainty of the various assignments. In this case, expected fuzzy utilities (or, in general, expected fuzzy values) are obtained. In the choice of alternatives, the problem arises of choosing the ordering between fuzzy numbers; moreover, a partial ordering of the alternatives is often obtained, since fuzzy numbers are not a totally ordered set with respect to the various orders other than the crisp order. A deepening of these aspects and an extension to games for people is in Mares (200).

4. A fuzzy partition of the territory for the construction of an alternative to rest homes

For each of the classes of seniors it is necessary to organize various types of services. To reduce travel and expenses it is proposed, for each service, to consider a *fuzzy partition* of the territory in microzones, starting from a *matrix of distances* and taking into consideration a suitable *exponent* of distances. The effectiveness of an urgent intervention can be compromised if the distances to be covered are high. To take this into account, we need to enhance the effect of distances by considering a high exponent. Similarly, if an intervention is expensive, one can try to reduce the cost only if the operator does many interventions in one day. This is possible only if he travels small distances. Also in this case the exponent of the distances must be high. In other words we can say that "the greater the urgency or the cost of the interventions, the greater the exponent". The control of the flow of services of a set of microzones must be entrusted to a *managerial figure* that has a relationship of trust and friendship with the elderly person.

The manager can be in a public body (for example, municipality or health district) or he can be a private administrator, he must solve all the bureaucratic problems related to *care and assistance* and must treat the *minimum price* with associations for nursing, catering, cleaning services, social assistance, companionship, etc.

In this way, on the one hand, these associations, working on a high number of elderly and in limited territory, will have a secure income, on the other the elderly will be immune to surprises and obstacles and will reduce his expenses (some summary calculations have brought us to believe that a saving of 80% can be achieved!).

If the manager is private, there may be a small expense for each elder for his compensation, much lower than the foreseeable expenses for obtaining the benefits without the help of the manager.

For the division in microzone our proposal is to follow the following algorithm, divided into three phases.

First phase, modeling of the observed situation

- step 1): fixed the territory T on which we operate, we identify a set G of g *geographic parameters* (for example the geographical coordinates or the travel time to move from one point to another of T);
- step 2): we identify a set K of k *criteria* related to the types of care or assistance needs of the elderly;
- step 3): we introduce in R^{g+k} a *distance* obtained as a linear combination with positive coefficients of the absolute values of differences of the coordinates;
- step 4): we assume that, on the basis of a careful medical-social investigation, every elder of the territory T is represented by a *point* of R^{g+k} in which the first g coordinates are of geographic type, and the other k coordinates are positive real numbers that represent the measures in which the elderly need the various types of care or assistance;
- step 5): called Ω the set of points of R^{g+k} representing elderly, we introduce, in R^{g+k}, a *weight function* $\lambda: R^{g+k} \to [0, +\infty)$ null in $R^{g+k} -\Omega$ and positive in Ω.

Second phase, of fuzzy partition of Ω

- step 1): a *fuzzy classification algorithm* is chosen, linked to the matrix of the distances between the points of Ω, to the weight function λ and to parameters c, f, s which represent, the *number of classes*, the *degree of fuzzification* and an *exponent of distances*, respectively;
- step 2): the *initial values* of the algorithm are chosen;
- step 3): the algorithm *is executed*.

Third phase, analysis and adjustment of the results obtained

- step 1): check if the cardinality of the obtained classes are *sufficiently close*;
- step 2): the classes are examined with politicians and experts of assistance and care for the elderly to see if there are no *practical difficulties*;
- step 3): if points 1) and 2) have not risen difficulties, we *accept* the fuzzy partition obtained, otherwise we decide a modification strategy that can be an *empirical modification* of the classes or a *modification of the a priori choices* in the first two phases.

If we make three fuzzy partitions Π_i, i = 1, 2, 3, one for each of the three fuzzy numbers u_i, that make up the fuzzy partition of the age of the elders, then, for each Π_i the weight λ_{ij} of each elder O_j must be proportional to $u_i(O_j)$, degree of belonging of O_j to class i.

5. Conclusions and research perspectives

Regarding the fuzzy classification algorithm, we can use the DMA, considered in (Ferri and Maturo 1999, 2001) which is a generalization of the algorithms of Bezdek (1981) and Ross (1997).

The algebraic-geometrical interpretation introduced in (Ferri and Maturo 1999, 2001), of the algebraic hyperstructures and geometric join spaces (Corsini 1993, 1994; Prenowitz and Jantosciak 1979), can be a useful complement to the comprehension of the results obtained.

Important results can be obtained using cooperative game theory that can increase the usefulness of all players considered in Sec 3 (see e.g. Von Neumann and Morgenstern 1947).

A complete discussion from the fuzzy cooperative game theory point of view can be made by applying the theories illustrated in Mares (2001), distinguishing between the possibilities of utility exchange or not.

Instead, starting from the point of view of decision theory, further results can be obtained by modeling the data as fuzzy events and using appropriate fuzzy measures (Maturo 2012).

Finally, modeling through game theory could be usefully replaced by the algorithms for *building consensus* among the various actors, according to the lines illustrated in various works, e.g. in (Maturo et. al. 2013; Maturo and Ventre 2012).

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